THE DIMENSION OF HYPERSPACES OF NON-METRIZABLE CONTINUA

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ABSTRACT. We prove that, for any Hausdorff continuum X, if $\dim X \geq 2$ then the hyperspace $C(X)$ of subcontinua of X is not a C-space; if $\dim X = 1$ and X is hereditarily indecomposable then $\dim C(X) = 2$ or $C(X)$ is not a C-space. This generalizes results known for metric continua.

1. Introduction

Throughout the paper all spaces are normal. A continuum is a compact, connected Hausdorff space. By dimension we always mean the covering dimension dim. A continuum X is hereditarily indecomposable iff for each subcontinua $A, B \subseteq X$ we have $A \subseteq B$, $B \subseteq A$ or $A \cap B = \emptyset$. For a compact X denote by $K(X)$ the hyperspace of all non-empty subcompacta of X, equipped with the Vietoris topology. By $C(X)$ we denote the hyperspace of all non-empty subcontinua of X, with the topology inherited from $K(X)$.

Definition 1.1. A space X is a C-space (or has property C) if and only if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$, such that each $\mathcal{V}_i$ is a family of pairwise disjoint open subsets of X, $\mathcal{V}_i \prec \mathcal{U}_i$ ($\mathcal{V}_i$ refines $\mathcal{U}_i$, i.e. $\forall\, V \in \mathcal{V}_i\ \exists U \in \mathcal{U}_i\ V \subseteq U$) and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of X.

We refer to [3] for basic properties of C-spaces. It is easy to observe that C-spaces are weakly infinite dimensional. The class of C-spaces contains finite dimensional spaces and countable dimensional metric spaces.

We prove the following theorem:

Theorem 1.2.

(i) *Suppose X is a continuum of dimension ≥ 2. Then $C(X)$ is not a C-space.*

(ii) *Suppose X is a 1-dimensional hereditarily indecomposable continuum. Then either $\dim C(X) = 2$ or $C(X)$ is not a C-space.*

The theorem is already known for metric continua. Part (i) was stated by M. Levin and J. T. Rogers, Jr. in [8]. Part (ii) can be obtained using methods from [2, 8, 9] (see [10, Theorem 3.1]).

To prove it for non-metric spaces we use the technique of lattices and Wallman representations as well as some set-theoretical methods, as it was done in [1]. We refer to [11] for the definition of a lattice and preliminary facts on Wallman spaces. We consider only distributive and separative lattices.

Date: September 18, 2012.

2010 *Mathematics Subject Classification.* Primary 54F45; Secondary 03C98, 54B20.

Key words and phrases. continuum, hyperspace, dimension, C-space, elementary submodel, Wallman space.

2. Lattices and Wallman spaces

For a compact space X we consider the lattice 2^X of closed subsets of X with $\cup$ and $\cap$ as lattice operations, $\emptyset$ and X as the minimal and maximal elements. Each lattice L corresponds to the Wallman space wL consisting of all ultrafilters on L. For $a \in L$ let $\widehat{a} = \{u \in wL \colon a \in u\}$. We define the topology in wL taking the family $\{\widehat{a} \colon a \in L\}$ as a base for closed sets.

It is easy to show that $w2^X$ is homeomorphic to X. More generally, the following fact holds true:

Fact 2.1. *If $\mathcal{F}$ is a base for closed sets in X which is closed under finite unions and intersections (so $\mathcal{F}$ is a lattice), then $w\mathcal{F}$ is homeomorphic to X.*

Proof. We define the homeomorphism $h \colon X \to w\mathcal{F}$ in the natural way: $h(x) = \{F \in \mathcal{F} \colon x \in F\}$. It is not difficult but tedious to verify that h is a well-defined homeomorphism indeed. We leave it as an exercise. □

Definition 2.2. A lattice L is normal iff

$$L \models \forall a, b \, (a \cap b = 0_L \to \exists c, d \, (c \cup d = 1_L \wedge c \cap a = 0_L \wedge d \cap b = 0_L))$$

We collect some well-known observations.

Fact 2.3 (see, e.g., [11])**.** *L is normal if and only if wL is Hausdorff.*

Fact 2.4 ([11, Theorem 2.6])**.** *If L is a countable normal lattice then wL is a compact metric space.*

Remark 2.5. *A sublattice L of L^* yields the continuous surjection $q \colon wL^* \to wL$, given by $q(u) = u \cap L$.*

3. Proof of Theorem 1.2

The proof is rather simple, but it uses some set-theoretic framework. We deal with some inner model of (large enough fragment of) ZFC and its countable elementary submodel.

Our strategy is to bring the non-metric case to the metric one. Suppose X is a non-metric continuum. We will find a countable sublattice $L \subseteq 2^X$ such that wL is a metric continuum, $\dim wL = \dim X$ and $\dim C(wL) = \dim C(X)$. Moreover, wL $[C(wL)]$ is hereditarily indecomposable if and only if such is X $[C(X)]$ and wL $[C(wL)]$ is a C-space if and only if such is X $[C(X)]$.

We apply the technique used in [1] to find the sublattice L.

For an infinite cardinal κ, $H(\kappa)$ is the set of all sets x, such that $|TC(x)| < \kappa$. (TC is the transitive closure, i.e. $TC(x) = x \cup \bigcup x \cup \bigcup\bigcup x \cup \ldots$). If κ is regular then $H(\kappa)$ is a model of ZFC without the Power Set Axiom (see [7, p. 162]). But if κ is large enough, then there are power sets in $H(\kappa)$ for all sets we need.

Let X be a (non-metric) continuum. Fix a suitably large regular cardinal κ (it is enough if $\mathcal{P}(\mathcal{P}(X)) \in H(\kappa)$). Take a countable elementary submodel $\mathcal{M} \prec H(\kappa)$, such that $X \in \mathcal{M}$ (use the Löwenheim-Skolem theorem). Then $\mathcal{M}$ also models enough of ZFC. Moreover, every finite subset of $\mathcal{M}$ belongs to $\mathcal{M}$. Denote $L = 2^X \cap \mathcal{M}$. By elementarity, L is a normal sublattice of 2^X. Since L is countable, applying Fact 2.4 and Remark 2.5, we obtain:

Fact 3.1. *wL is a metric continuum.* □

Let us recall two well-known facts.

Proposition 3.2 (see [5, Subsection 4.1])**.** $\dim X = \dim wL$. *More generally, let K^* be a lattice in $\mathcal{M}$ and $K = K^* \cap \mathcal{M}$. Then $\dim wK^* = \dim wK$.*

Proposition 3.3. *A continuum X is hereditarily indecomposable if and only if such is wL.*

The *if* part is straightforward. For the *only if* see [6, Lemma 2.2].

Now we prove a similar fact about property C.

Theorem 3.4. *The space X is a C-space if and only if so is wL. More generally, let K^* be a lattice in $\mathcal{M}$ and $K = K^* \cap \mathcal{M}$. Then wK^* is a C-space if and only if such is wK.*

Proof. We provide the proof for the first part of the proposition. It can be easily adopted for the more general statement.

Denote $\mathcal{B} = \{wL \setminus \widehat{F}\colon\ F \in L\}$ (the open base for wL, which is closed under finite unions and intersections).

($\Leftarrow$) We will show that if X is not a C-space then neither is wL. Assume X is not a C-space. Then, by compactness there exists a sequence $(\mathcal{U}_i)_{i=1}^{\infty}$ of finite open covers of X, such that for every $m \geq 1$ and finite families of open disjoint sets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m$ which satisfy $\mathcal{V}_i \prec \mathcal{U}_i$, their union $\mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots \cup \mathcal{V}_m$ is not a cover of X (compactness allows to consider only finite families). Translating it into terms of lattice 2^X we obtain that $H(\kappa)$ models the following sentence φ:

(φ) There exists a sequence $(\mathcal{F}_i)_{i=1}^{\infty}$ of finite subsets of 2^X such that for each $i \geq 1$ the intersection $\bigcap \mathcal{F}_i$ is empty and for every $m \geq 1$ and finite $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m \subseteq 2^X$ the following holds:

($*$) If for each $j \leq m$ and $G \in \mathcal{G}_j$ there exists $F \in \mathcal{F}_j$ such that $F \subseteq G$ and for any distinct $G, G' \in \mathcal{G}_j$ we have $G \cup G' = X$, then $\bigcap(\mathcal{G}_1 \cup \mathcal{G}_2 \cup \ldots \cup \mathcal{G}_m) \neq \emptyset$.

$\mathcal{M} \models \varphi$ by elementarity. So there is $(\mathcal{F}_i)_{i=1}^{\infty} \in \mathcal{M}$ as in φ, such that ($*$) holds for every $m < \omega$ and $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m \in \mathcal{M}$.

The sequence $(\mathcal{F}_i)_{i=1}^{\infty}$ gives rise to a sequence $(\mathcal{U}_i)_{i=1}^{\infty}$ of open covers of wL (namely $\mathcal{U}_i = \{wL \setminus \widehat{F}\colon F \in \mathcal{F}_i\}$), which witnesses that wL is not a C-space. Indeed, suppose we have a finite sequence $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m$ of finite families of open disjoint sets, $\mathcal{V}_i \prec \mathcal{U}_i$ and their union is a cover of wL. We can produce $\mathcal{V}'_1, \mathcal{V}'_2, \dots, \mathcal{V}'_m$, which are additionally contained in the base $\mathcal{B}$: shrink each $V \in \bigcup_{i=1}^m \mathcal{V}_i$ to a closed set C_V so that $\bigcup_{i=1}^m \{C_V\colon V \in \mathcal{V}_i\}$ forms a closed cover of wL. Since C_V is compact, it can be covered by finitely many sets $B_1^V, B_2^V, \dots, B_{j(V)}^V \subseteq V$ from the basis $\mathcal{B}$. Let $V' = B_1^V \cup B_2^V \cup \ldots \cup B_{j(V)}^V$. We have $V' \in \mathcal{B}$, since $\mathcal{B}$ is closed under finite unions. Define $\mathcal{V}'_i = \{V'\colon V \in \mathcal{V}_i\}$.

Having $\mathcal{V}'_1, \mathcal{V}'_2, \dots, \mathcal{V}'_m$ it is easy to get $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_m \in \mathcal{M}$ which do not satisfy ($*$). Indeed, each $V' \in \mathcal{V}'_i$ is given by some $F_{V'} \in L$ via $V' = wL \setminus \widehat{F}_{V'}$. Then $\mathcal{G}_i = \{F_{V'}\colon V' \in \mathcal{V}'_i\}$. Since $\mathcal{V}'_i \subseteq \mathcal{B}$, we have $\mathcal{G}_i \in \mathcal{M}$.

($\Rightarrow$) Suppose $\mathcal{U}_1, \mathcal{U}_2, \dots$ is a sequence of finite open covers of wL, say $\mathcal{U}_i = \{U_{i1}, U_{i2}, \dots, U_{ik_i}\}$. Without loss of generality we may assume that each $\mathcal{U}_i$ consists

of sets from $\mathcal{B}$, i.e. for each $i \in \mathbb{N}$ and $j \leq k_i$ there is some $F_{ij} \in \mathcal{M}$ closed in X such that $U_{ij} = wL \setminus \widehat{F_{ij}}$.

Define $U'_{ij} = X \setminus F_{ij}$ and $\mathcal{U}'_i = \{U'_{i1}, U'_{i2}, \ldots, U'_{ik_i}\}$. Note that $\mathcal{U}'_i$ is an open cover of X since $F_{i1} \cap F_{i2} \cap \ldots \cap F_{ik_i} = \emptyset$ ($\mathcal{U}_i$ is a cover of wL).

Since X is a compact C-space there exist $n \in \mathbb{N}$ and finite families of pairwise disjoint open sets $\mathcal{V}'_1, \mathcal{V}'_2, \ldots, \mathcal{V}'_n$ such that each $\mathcal{V}'_i$ refines $\mathcal{U}'_i$ and $\bigcup_{i=1}^n \mathcal{V}'_i$ is a cover of X. Let us code this in terms of the lattice 2^X. First denote $\mathcal{V}'_i = \{V'_{i1}, V'_{i2}, \ldots, V'_{il_i}\}$ and $G'_{ij} = X \setminus V'_{ij}$ for $i \leq n$ and $j \leq l_i$. The following sentence ψ is true in $H(\kappa)$:

$$(\psi) \begin{cases} \text{There exist } G'_{11}, G'_{12}, \ldots, G'_{1l_1}, G'_{21}, G'_{22}, \ldots G'_{2l_2}, \ldots, G'_{n1}, G'_{n2}, \ldots, G'_{nl_n} \\ \text{such that:} \\ (1)\ \bigwedge_{i=1}^n \left(\bigwedge_{1 \leq j < j' \leq l_i} (G'_{ij} \cup G'_{ij'} = X) \right) \\ (2)\ \bigwedge_{i=1}^n \left(\bigwedge_{j=1}^{l_i} \left(\bigvee_{j'=1}^{k_i} (G'_{ij} \cap F_{ij'} = F_{ij'}) \right) \right) \\ (3)\ \bigcap_{i=1}^n \bigcap_{j=1}^{l_i} G'_{ij} = \emptyset. \end{cases}$$

Symbols $\bigwedge$ and $\bigvee$ abbreviate finite conjuctions and disjunctions. Note that F_{ij}'s appear in ψ as parameters from $\mathcal{M}$.

We have $H(\kappa) \models \psi$ and by elementarity $\mathcal{M} \models \psi$. Hence, for $i \leq n$ and $j \leq l_i$ there are $G_{ij} \in \mathcal{M}$ which satisfy $(1-3)$ when placed in ψ instead of G'_{ij}. Take $V_{ij} = wL \setminus \widehat{G_{ij}}$ and $\mathcal{V}_i = \{V_{i1}, V_{i2}, \ldots, V_{il_k}\}$. Then $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ are families of pairwise disjoint sets (by (1)), open in wL. For $i \leq n$ the family $\mathcal{V}_i$ refines $\mathcal{U}_i$ (by (2)) and $\bigcup_{i=1}^n \mathcal{V}_i$ is a cover of wL (by (3)). □

Now we will link the space X with its hyperspace $C(X)$ in terms of lattices. Namely, having the lattice 2^X we define a lattice $K^* \in \mathcal{M}$, such that wK^* is homeomorphic to $C(X)$. Then, taking $K = K^* \cap \mathcal{M}$ we will show that wK is homeomorphic to $C(wL)$.

$C(X)$ is defined (as a set) only in terms of 2^X:

$$C(X) = \{F \in 2^X : \neg(\exists G_1, G_2 \in 2^X)(G_1 \cup G_2 = F \wedge G_1 \cap G_2 = \emptyset)\}.$$

Define K^* as a sublattice of $(\mathcal{P}(C(X)), \cup, \cap, \emptyset, C(X))$ generated by the family $\{\mathcal{F}^* : \mathcal{F} \in [2^X]^{<\omega}\}$, where

$$\mathcal{F}^* = C(X) \setminus \{G \in C(X) : G \cap \bigcap \mathcal{F} = \emptyset \wedge (\forall F \in \mathcal{F})(F \cup G \neq F)\}.$$

The lattice K^* is the closure under finite unions and intersections of the family of sets $\mathcal{F}^*$ for all finite $\mathcal{F} \subseteq 2^X$. It is easy to verify that sets $\mathcal{F}^*$ form a closed base for $C(X)$. Hence, K^* is a closed base and a lattice simultaneously. By Fact 2.1, we get:

Remark 3.5. *$C(X)$ is homeomorphic to wK^*.*

Since $X \in \mathcal{M}$, it follows directly by the definition of K^* that $K^* \in \mathcal{M}$. Take $K = K^* \cap \mathcal{M}$. The only thing we still lack is:

Proposition 3.6. *wK is homeomorphic to $C(wL)$.*

Proof. We know that K^* is generated by the family $\{\mathcal{F}^* : \mathcal{F} \in [2^X]^{<\omega}\}$. By elementarity, K is generated by $\{\mathcal{F}^* : \mathcal{F} \in [L]^{<\omega}\}$. Note that a basic closed set in $C(wL)$ is determined by $\mathcal{F} \in [L]^{<\omega}$ via the formula

$$\mathcal{C}_{\mathcal{F}} = C(wL) \setminus \{C \in C(wL) : C \cap \bigcap \{\widehat{F} : F \in \mathcal{F}\} = \emptyset \wedge (\forall F \in \mathcal{F})(\widehat{F} \cup C \neq \widehat{F})\}$$

(since L is isomorphic to a closed base for wL). Hence, the lattice K is isomorphic to the lattice generated by $\{\mathcal{C}_{\mathcal{F}} \colon \mathcal{F} \in [L]^{<\omega}\}$, which forms a closed base for $C(wL)$. By Fact 2.1 wK is homeomorphic to $C(wL)$. $\square$

Now we have all ingredients to prove Theorem 1.2.

Proof of Theorem 1.2.

(i) Suppose that $\dim X \geq 2$. Proposition 3.2 gives $\dim wL \geq 2$. By the results of M. Levin, J. T. Rogers, Jr. for metric continua [8] we have that $C(wL)$ is not a C-space. But $C(wL)$ is homeomorphic to wK (Proposition 3.6). Hence wK^* is neither a C-space (Theorem 3.4). By Remark 3.5, $C(X)$ is homeomorphic to wK^*, so it is not a C-space.

(ii) Similarily, suppose that X is a 1-dimensional, hereditarily indecomposable continuum. Then wL is also 1-dimensional (Proposition 3.2) and hereditarily indecomposable (Proposition 3.3). By result known for metric continua ([10, Theorem 3.1]) we have that $C(wL)$ is either 2-dimensional or is not a C-space. By Proposition 3.6, $C(wL)$ is homeomorphic to wK. Therefore, wK^* is either 2-dimensional (Proposition 3.2) or is not a C-space (Theorem 3.4). But wK^* is homeomorphic to $C(X)$ by Remark 3.5. $\square$

4. Remarks on m-C-spaces

Definition 4.1 ([4])**.** For $m \geq 2$ a space X is said to be an m-C-space if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of m-element open covers of X, there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$, such that each $\mathcal{V}_i$ is a family of pairwise disjoint open subsets of X, $V_i \prec \mathcal{U}_i$ and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of X.

Observe that

$$2\text{-}C\text{-spaces} \supseteq 3\text{-}C\text{-spaces} \supseteq \ldots \supseteq m\text{-}C\text{-spaces} \supseteq \ldots \supseteq C\text{-spaces}.$$

Moreover, the following holds

Fact 4.2 ([4, Proposition 2.11])**.** *A space is weakly infinite dimensional if and only if it is a 2-C-space.*

One can easily adopt the proof of Theorem 3.4 to obtain the following:

Proposition 4.3. *Let K^* be a lattice in $\mathcal{M}$ and $K = K^* \cap \mathcal{M}$. Then wK^* is an m-C-space if and only if such is wK.* $\square$

Let us recall two definitions and one question from [11]:

Definition 4.4 ([11, Definition 2.7])**.** We will say that a property $\mathcal{P}$ of a compact space is *elementarily reflected* if whenever some compact space X has the property $\mathcal{P}$ then the Wallman representation wL of any elementary sublattice L of 2^X also has property $\mathcal{P}$.

Definition 4.5 ([11, Definition 2.8])**.** A property $\mathcal{P}$ of a compact space is *elementarily reflected by submodels* if whenever some compact space X has the property $\mathcal{P}$ then the Wallman representation wL of any elementary sublattice L of the form $L = 2^X \cap \mathcal{M}$, where $2^X \in \mathcal{M}$ and $\mathcal{M} \prec H(\kappa)$ (for a large enough regular κ), also has property $\mathcal{P}$.

Question 4.6 ([11, Question 2.32])**.** *Is having strong infinite dimension elementarily reflected, and is having not strong infinite dimension elementarily reflected?*

Recall that, by definition, a space is strongly infinite dimensional if it is not weakly infinite dimensional.

Proposition 4.3 gives a partial answer to this question. Indeed, in particular it says that that both these properties are elementarily reflected by submodels (use the characterization of the weak infinite dimension from Fact 4.2). Moreover, following the proof of the Theorem 3.4 one can observe that the model $\mathcal{M} \prec H(\kappa)$ is not needed for the left-to-right implication. That means property C is elementarily reflected and the opposite is elementarily reflected by submodels. Properties m-C and non-m-C behave in the same way. Summarizing, we can say that having strong infinite dimension is elementarily reflected by submodels, and having not strong infinite dimension is elementarily reflected.

It is not known if the notions of property C and weakly infinite dimension coincide within the class of compact spaces. However, since both properties are elementarily reflected by submodels, there exists a metric counterexample which distinguishes these two notions if and only if there exists a non-metric one.

Acknowledgement. The author is indebted to Piotr Borodulin-Nadzieja, Ahmad Farhat and Paweł Krupski for helpful discussions.

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